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Lagrange Type Errors for Truncated Jacobi Series

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Let s_n denote the formal expansion of a function f in a Jacobi series truncated after $n+1$ terms. For $f \in C^{n+1}[-1, 1]$ the uniform norm of $f - s_n$ is expressed in terms of the $(n+1)$th derivative of f. This expression is precise when $\max(\alpha, \beta) \geqslant -\frac{1}{2}$ and when $-1 < \alpha = \beta < -\frac{1}{2}$ with n odd. For other values of α and β an asymptotic expression for the norm is derived. Comparisons are made with the minimax polynomial of degree no greater than n, for which it is known that $\|f - p_n\|_\infty = (2^n(n+1)!)^{-1} |f^{(n+1)}(\eta)|$ for some $\eta \in (-1, 1)$.

1. INTRODUCTION

Suppose that $f \in C^{n+1}[-1, 1]$ and f is approximated by s_n, the $(n+1)$th partial sum of the Jacobi series of f, given by

$$s_n(x) = \sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(x), \tag{1.1}$$

where

$$a_k = h_k^{-1} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta f(t) P_k^{(\alpha,\beta)}(t)\, dt, \tag{1.2}$$

and

$$h_k = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta (P_k^{(\alpha,\beta)}(t))^2\, dt$$
$$= \frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1)\, \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\, \Gamma(k+\alpha+\beta+1)\, k!}, \tag{1.3}$$

with $\alpha > -1$ and $\beta > -1$.

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The remainder term of the approximation, $r_n(x)$, is given by

$$r_n(x) = f(x) - s_n(x). \tag{1.4}$$

The aim of this paper is to derive an expression for $\|r_n\|_\infty$ in terms of the $(n+1)$th derivative of f, where $\|r_n\|_\infty = \max_{-1 \leq x \leq 1} |r_n(x)|$.

For the minimax polynomial of degree no greater than n, p_n say, which approximates f on $[-1, 1]$, Bernstein [2] has shown that

$$\|f - p_n\|_\infty = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\xi)|, \tag{1.5}$$

for some $\xi \in (-1, 1)$.

Light [5] has investigated bounds for the norm of s_n, regarded as a linear operator on $C[-1, 1]$. His results tend to support the numerical evidence that, for each n and $\alpha, \beta \geq -\frac{1}{2}$, the minimum of $\|s_n\|$ is attained when $\alpha = \beta = -\frac{1}{2}$. A similar result is obtained in this paper for $\|r_n\|_\infty$, in that, the form (1.5) is obtained when $\alpha = \beta = -\frac{1}{2}$, otherwise the coefficient of $|f^{(n+1)}(\xi)|$ is greater than $1/(2^n(n+1)!)$.

The first step in the analysis is to derive an expression for $r_n(x)$ as an integral whose integrand contains a linear factor $g_x(t)$ (see (2.12)). Treatment of this integral depends upon whether x is such that $g_x(t)$ is of constant sign on $(-1, 1)$ or $g_x(t)$ has a zero on $(-1, 1)$. From the first case a lower bound for $\|r_n\|_\infty$ is obtained. An upper bound for $\|r_n\|_\infty$ is obtained from both cases. When $\max(\alpha, \beta) \geq -\frac{1}{2}$ or $-1 < \alpha = \beta < -\frac{1}{2}$ with n odd, these two results are combined to give an expression of the form

$$\|r_n\|_\infty = d_n |f^{(n+1)}(\xi)|. \tag{1.6}$$

For other values of α and β a similar asymptotic formula, for large n, is derived.

When $s_n(x)$ is the truncated Chebyshev series of the first kind $(\alpha = \beta = -\frac{1}{2})$ then $d_n = 1/(2^n(n+1)!)$, the same as for the minimax polynomial.

A surprising result is that when $-1 < \alpha, \beta < -\frac{1}{2}$ then the coefficient d_n of the asymptotic formula is $K/(2^n(n+1)!)$, where K is some constant between 1 and $\sqrt{2}$.

2. The Remainder $r_n(x)$

Substitute a_k from (1.2) into (1.1) and interchange the order of summation and integration to give

$$s_n(x) = \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta f(t) \sum_{k=0}^{n} h_k^{-1} P_k^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(x)\, dt. \tag{2.1}$$

From the orthogonality property of the Jacobi polynomials it is clear that for $f(t) \equiv 1$ the above integral is 1. It follows that

$$r_n(x) = \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}(f(x)-f(t)) \sum_{k=0}^{n} h_k^{-1} P_k^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(x)\, dt. \tag{2.2}$$

The Christoffel–Darboux formula for Jacobi polynomials (see [1]) is

$$\sum_{k=0}^{n} h_k^{-1} P_k^{(\alpha,\beta)}(t)\, P_k^{(\alpha,\beta)}(x) = A\, \frac{P_{n+1}^{(\alpha,\beta)}(t)\, P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(t)\, P_{n+1}^{(\alpha,\beta)}(x)}{t-x}, \tag{2.3}$$

where

$$A = \frac{(n+1)!\, \Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta}(2n+\alpha+\beta+2)\, \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}. \tag{2.4}$$

Substitution from (2.3) into (2.2) gives

$$r_n(x) = A \int_{-1}^{1} \frac{f(t)-f(x)}{t-x} (1-t)^{\alpha}(1+t)^{\beta}[P_n^{(\alpha,\beta)}(t)\, P_{n+1}^{(\alpha,\beta)}(x) - P_{n+1}^{(\alpha,\beta)}(t)\, P_n^{(\alpha,\beta)}(x))]\, dt. \tag{2.5}$$

Rodriguez' formula for Jacobi polynomials (see [1]) is

$$(1-t)^{\alpha}(1+t)^{\beta}\, P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n} [(1-t)^{n+\alpha}(1+t)^{n+\beta}]. \tag{2.6}$$

Hence

$$(1-t)^{\alpha}(1+t)^{\beta}[P_n^{(\alpha,\beta)}(t)\, P_{n+1}^{(\alpha,\beta)}(x) - P_{n+1}^{(\alpha,\beta)}(t)\, P_n^{(\alpha,\beta)}(x)] = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n} [(1-t)^{n+\alpha}(1+t)^{n+\beta} g_x(t)], \tag{2.7}$$

where

$$g_x(t) = P_{n+1}^{(\alpha,\beta)}(x) + \frac{\beta-\alpha-t(2n+\alpha+\beta+2)}{2(n+1)}\, P_n^{(\alpha,\beta)}(x). \tag{2.8}$$

Observe that

$$\frac{f(t)-f(x)}{t-x} = \int_0^1 f'((t-x)\, u + x)\, du. \tag{2.9}$$

On substituting (2.7) and (2.9) into (2.5) and interchanging the order of integration, we have

$$r_n(x)=\frac{(-1)^n A}{2^n n!}\int_0^1\left(\int_{-1}^1 f'((t-x)\,u+x)\right.$$
$$\left.\times\frac{d^n}{dt^n}\left[(1-t)^{n+\alpha}(1+t)^{n+\beta}g_x(t)\right]dt\right)du. \tag{2.10}$$

Since, for $1\leqslant k\leqslant n$,

$$\frac{d^{n-k}}{dt^{n-k}}\left[(1-t)^{n+\alpha}(1+t)^{n+\beta}g_x(t)\right]_{t=\pm 1}=0,$$

the integral with respect to t, in (2.10), may be integrated by parts n times to give

$$r_n(x)=\frac{A}{2^n n!}\int_0^1 u^n\left(\int_{-1}^1 f^{(n+1)}((t-x)\,u+x)(1-t)^{n+\alpha}(1+t)^{n+\beta}g_x(t)\,dt\right)du. \tag{2.11}$$

Applying the mean value theorem to the integral with respect to u we have

$$r_n(x)=\frac{A}{2^n(n+1)!}\int_{-1}^1 f^{(n+1)}((t-x)\,u_0+x)(1-t)^{n+\alpha}(1+t)^{n+\beta}g_x(t)\,dt, \tag{2.12}$$

for some $u_0\in(0,1)$. This is the required integral form for $r_n(x)$.

If $g_x(t)$ is of constant sign on $(-1,1)$ then application of the mean value theorem is straightforward and an expression for $r_n(x)$ is obtainable. However, if $g_x(t)$ changes sign on $(-1,1)$ then we can only obtain an upper bound for $|r_n(x)|$. The two cases are treated separately.

THEOREM 2.1. *If x is such that*

$$-(n+1)/(n+\beta+1)\leqslant P_n^{(\alpha,\beta)}(x)/P_{n+1}^{(\alpha,\beta)}(x)\leqslant(n+1)/(n+\alpha+1),$$

then, for some $\eta\in(-1,1)$,

$$r_n(x)=\frac{2^{n+1}\Gamma(n+\alpha+\beta+2)}{\Gamma(2n+\alpha+\beta+3)}P_{n+1}^{(\alpha,\beta)}(x)\,f^{(n+1)}(\eta). \tag{2.13}$$

Proof. The condition on x implies that $g_x(t)$ is of constant sign on $(-1,1)$. This is readily seen by observing that for such x, $g_x(-1)\,g_x(1)\geqslant 0$

and $g_x(t)$ is linear in t. Thus applying the mean value theorem to (2.12) gives

$$r_n(x)=\frac{Af^{(n+1)}(\eta)}{2^n(n+1)!}\int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}g_x(t)\,dt, \tag{2.14}$$

for some $\eta\in(-1,1)$. Writing $g_x(t)$ as

$$g_x(t)=(1-t)\,g_x(-1)/2+(1+t)\,g_x(1)/2,$$

and using the formula (derived from [1; 6.2.1]),

$$\int_{-1}^{1}(1-t)^a(1+t)^b\,dt=\frac{2^{a+b+1}\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}, \tag{2.15}$$

the integral of (2.14) may be evaluated to give the result (2.13). ∎

THEOREM 2.2. *If* $q=\max(\alpha,\beta)$ *and* x *is such that*

$$-(n+\beta+1)/(n+1)<P_{n+1}^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(x)<(n+\alpha+1)/(n+1),$$

then for some $\eta\in(-1,1)$,

$$|r_n(x)|<\frac{2^{n+1}\Gamma(n+\alpha+\beta+2)}{\Gamma(2n+\alpha+\beta+3)}\left(\frac{n+q+1}{n+1}\right)|P_n^{(\alpha,\beta)}(x)|\,|f^{(n+1)}(\eta)|. \tag{2.16}$$

Proof. The condition on x implies that $g_x(t)$ changes sign on $(-1,1)$. This is readily seen by observing that for such x, $g_x(-1)\,g_x(1)<0$, and $g_x(t)$ is linear in t. From (2.12) we have

$$|r_n(x)|\leqslant\frac{A}{2^n(n+1)!}\int_{-1}^{1}|f^{(n+1)}((t-x)\,u_0+x)|(1-t)^{n+\alpha}(1+t)^{n+\beta}|g_x(t)|\,dt.$$

Applying the mean value theorem to this integral gives

$$|r_n(x)|\leqslant\frac{A}{2^n(n+1)!}|f^{(n+1)}(\eta)|\int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}|g_x(t)|\,dt, \tag{2.17}$$

for some $\eta\in(-1,1)$. We note that, for $t\in(-1,1)$,

$$|g_x(t)|<(1-t)|g_x(-1)|/2+(1+t)|g_x(1)|/2.$$

Hence, using (2.15), we have

$$\begin{aligned}\int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}|g(t)|\,dt\\ &<B((n+\alpha+1)|g_x(-1)|+(n+\beta+1)|g_x(1)|),\\ &\leqslant B(n+q+1)(|g_x(-1)|+|g_x(1)|),\end{aligned} \tag{2.18}$$

where

$$B = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+3)}. \tag{2.19}$$

Since $g_x(-1)$ and $g_x(1)$ are of opposite sign, then

$$|g_x(-1)| + |g_x(1)| = |g_x(-1) - g_x(1)| = \frac{2n+\alpha+\beta+2}{n+1}\,|P_n^{(\alpha,\beta)}(x)|. \tag{2.20}$$

Combining the results (2.17) to (2.20) we recover the required inequality (2.16). ∎

We now use these two theorems to obtain uniform bounds for $|r_n(x)|$ on $[-1, 1]$ and, where possible, expressions for the uniform norm of r_n.

3. Uniform Bounds for $|r_n(x)|$

To obtain uniform bounds for $|r_n(x)|$ from Theorems 2.1 and 2.2 we first require bounds for $|P_n^{(\alpha,\beta)}(x)|$ on $[-1, 1]$. Such bounds are given by, for example, Szegö [7; Chap. 7] and Erdélyi [4; 10.18]. The form of these bounds depends upon the values of α and β:

For $q = \max(\alpha, \beta) \geqslant -\frac{1}{2}$,

$$|P_n^{(\alpha,\beta)}(x)| \leqslant \max((-1)^n P_n^{(\alpha,\beta)}(-1), P_n^{(\alpha,\beta)}(1)) = \frac{\Gamma(n+q+1)}{\Gamma(q+1)\,n!}; \tag{3.1}$$

for $-1 < \alpha = \beta < -\frac{1}{2}$,

$$|P_n^{(\alpha,\beta)}(x)| \leqslant |P_n^{(\alpha,\alpha)}(0)| = \frac{\Gamma(n+\alpha+1)}{2^n\Gamma(n/2+\alpha+1)(n/2)!} \quad \text{if } n \text{ is even,} \tag{3.2i}$$

$$|P_n^{(\alpha,\beta)}(x)| \leqslant P_n^{(\alpha,\alpha)}(x_n')| < \frac{\sqrt{(n+2\alpha+1)/n}\;\Gamma(n+\alpha+1)}{2^n\Gamma(n/2+\alpha+3/2)((n-1)/2)!} \quad \text{if } n \text{ is odd;} \tag{3.2ii}$$

for $-1 < \alpha \neq \beta < -\frac{1}{2}$,

$$|P_n^{(\alpha,\beta)}(x)| \leqslant |P_n^{(\alpha,\beta)}(x_n')|, \tag{3.3}$$

where x_n' is one of the two maximum points of $|P_n^{(\alpha,\beta)}(x)|$ closest to $(\beta-\alpha)/(\alpha+\beta+1)$.

Let m and M be defined by

$$m = \min_{-1 \leqslant x \leqslant 1} |f^{(n+1)}(x)|, \tag{3.4i}$$

$$M = \max_{-1 \leqslant x \leqslant 1} |f^{(n+1)}(x)|. \tag{3.4ii}$$

We are now ready to prove the main results of this paper.

THEOREM 3.1. *If* $q = \max(\alpha, \beta) \geq -\frac{1}{2}$ *then*

$$|r_n(x)| \leq \frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+q+2)}{(n+1)!\,\Gamma(2n+\alpha+\beta+3)\,\Gamma(q+1)} M. \tag{3.5}$$

Proof. If x is such that $g_x(t)$, given by (2.8), is of constant sign on $(-1, 1)$ then the result follows from Theorem 2.1 and the inequality (3.1). Equality may be achieved only if $f^{(n+1)}$ is constant on $(-1, 1)$ (otherwise, in Theorem 2.1, $m < |f^{(n+1)}(\eta)| < M$).

If x is such that $g_x(t)$ changes sign on $(-1, 1)$ then the result follows from Theorem 2.2 and the inequality (3.1). The inequality (3.5) is strict in this case. ∎

COROLLARY 3.1. *If* $q = \max(\alpha, \beta) \geq -\frac{1}{2}$ *then, for some* $\eta \in (-1, 1)$,

$$\|r_n\|_\infty = \frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+q+2)}{(n+1)!\,\Gamma(2n+\alpha+\beta+3)\,\Gamma(q+1)} |f^{(n+1)}(\eta)|. \tag{3.6}$$

Proof. Since $P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(\alpha+1)\,n!$ and $P_n^{(\alpha,\beta)}(1)/P_{n+1}^{(\alpha,\beta)}(1) = (n+1)/(n+\alpha+1)$, then, from Theorem 2.1,

$$|r_n(1)| \geq \frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+\alpha+2)}{(n+1)!\,\Gamma(2n+\alpha+\beta+3)\,\Gamma(\alpha+1)} m.$$

Similarly,

$$|r_n(-1)| \geq \frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+\beta+2)}{(n+1)!\,\Gamma(2n+\alpha+\beta+3)\,\Gamma(\beta+1)} m.$$

Thus

$$\|r_n\|_\infty = \max_{-1 \leq x \leq 1} |r_n(x)| \geq \frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+q+2)}{(n+1)!\,\Gamma(2n+\alpha+\beta+3)\,\Gamma(q+1)} m. \tag{3.7}$$

Using the same argument as in Theorem 3.1, equality in (3.7) may be achieved only if $f^{(n+1)}$ is constant on $(-1, 1)$. If $f^{(n+1)}$ is not constant on $(-1, 1)$ then the inequalities (3.5) and (3.7) are strict inequalities and the result (3.6) is valid for some $\eta \in (-1, 1)$. On the other hand if $f^{(n+1)}$ is constant on $(-1, 1)$ then $m = f^{(n+1)}(\eta) = M$ for all $\eta \in (-1, 1)$ and (3.6) follows immediately. ∎

An immediate consequence of this corollary is that if $s_n(x)$ is the truncated Chebyshev series ($\alpha = \beta = -\frac{1}{2}$) then, for some $\eta \in (-1, 1)$,

$$\|r_n\|_\infty = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\eta)|,$$

which duplicates the result (1.5) for the minimax polynomial. This particular result has recently been given in [3], where the method was also extended to other types of Chebyshev series.

THEOREM 3.2. *If* $-1<\alpha=\beta<-\frac{1}{2}$ *and* $N=[(n+1)/2]$[1] *then, for n odd,*

$$|r_n(x)| \leqslant \frac{\Gamma(N+\alpha+1/2)}{2^{n+1}N!\,\Gamma(n+\alpha+3/2)} M, \tag{3.8}$$

and, for n even,

$$|r_n(x)| < \frac{\Gamma(N+\alpha+1/2)}{2^{n+1}N!\,\Gamma(n+\alpha+3/2)} \frac{N+\alpha+1/2}{\sqrt{(N+\alpha+1)(N+1/2)}} M. \tag{3.9}$$

Proof. Both of these results follow from Theorems 2.1 and 2.2, the inequalities (3.2), and the duplication formula for the gamma function [1; 6.1.18],

$$\Gamma(2p)=\pi^{-1/2}2^{2p-1}\Gamma(p)\,\Gamma(p+1/2). \tag{3.10}$$

The dependence on x and the possible change of sign of $g_x(t)$ on $(-1, 1)$ is overcome by observing that, for $-1<\alpha<-\frac{1}{2}$,

$$\frac{N+\alpha+1/2}{N+1/2} < \frac{N+\alpha+1/2}{\sqrt{(N+\alpha+1)(N+1/2)}} < \sqrt{(N+\alpha)/(N-1/2)} < 1.$$

As a consequence we note that (3.8) is valid for all n. Also, we again note that equality may occur in (3.8) only if $f^{(n+1)}$ is constant on $[-1, 1]$. ▮

COROLLARY 3.2. *If* $-1<\alpha=\beta<-\frac{1}{2}$ *and n is odd then, for some* $\eta\in(-1, 1)$,

$$\|r_n\|_\infty = \frac{\Gamma(n/2+\alpha+1/2)}{2^{n+1}((n+1)/2)!\,\Gamma(n+\alpha+3/2)} |f^{(n+1)}(\eta)|. \tag{3.11}$$

Proof. For $-1<\alpha<-\frac{1}{2}$ and n odd the maximum of $|P_{n+1}^{(\alpha,\alpha)}(x)|$ on $[-1, 1]$ is attained at $x=0$ (see (3.2i)), so that

$$\max_{-1\leqslant x\leqslant 1} |P_{n+1}^{(\alpha,\alpha)}(x)| = |P_{n+1}^{(\alpha,\alpha)}(0)| = \frac{\Gamma(n+\alpha+2)}{2^{n+1}((n+1)/2)!\,\Gamma(n/2+\alpha+3/2)},$$

whilst $P_n^{(\alpha,\alpha)}(0)=0$. Therefore, from Theorem 2.1,

$$\|r_n\|_\infty \geqslant |r_n(0)| \geqslant \frac{\Gamma(n/2+\alpha+1)}{2^{n+1}((n+1)/2)!\,\Gamma(n+\alpha+3/2)} M, \tag{3.12}$$

[1] $[a]$ denotes the greatest integer not greater than a.

with equality possible only if $f^{(n+1)}$ is constant. The result (3.11) follows from (3.8) with $N=(n+1)/2$ and (3.12). ▮

We have been unable to establish a similar result for even values of n. For all other values of α and β we have

THEOREM 3.3. *If* $-1<\alpha, \beta<-\frac{1}{2}$ *and* $q=\max(\alpha, \beta)$ *then*

$$|r_n(x)| \leqslant 2^{n+1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(2n+\alpha+\beta+3)} D_n M, \tag{3.13}$$

where

$$D_n = \max\left\{|P_{n+1}^{(\alpha,\beta)}(x'_{n+1})|, \frac{n+q+1}{n+1}\,|P_n^{(\alpha,\beta)}(x'_n)|\right\}.$$

Proof. The result follows immediately from Theorems 2.1 and 2.2 and the inequality (3.3). ▮

Further results are possible utilising the asymptotic behaviour of $|P_n^{(\alpha,\beta)}(x)|$ for large n. In the notation of Olver [6], if $\lim_{n\to\infty} f(n)/\phi(n)=1$ then $f(n)\sim\phi(n)$, which reads, f is asymptotic to ϕ.

From Szegö [7, Chap. 8] we have, if $-1<\alpha, \beta<-\frac{1}{2}$ then

$$|P_n^{(\alpha,\beta)}(x'_n)| \sim \pi^{-1/2} 2^{-(\alpha+\beta+1)} K(\alpha, \beta)\, n^{-1/2}, \tag{3.14}$$

where x'_n is the maximum point of $|P_n^{(\alpha,\beta)}(x)|$ on $[-1, 1]$ and

$$K(\alpha, \beta) = |\alpha+1/2|^{-\alpha/2-1/4} |\beta+1/2|^{-\beta/2-1/4} |(\alpha+\beta+1)/4|^{(\alpha+\beta+1)/2}. \tag{3.15}$$

COROLLARY 3.3. *If* $-1<\alpha, \beta<-\frac{1}{2}$ *and* n *is large then, for some* $\eta\in(-1, 1)$,

$$\|r_n\|_\infty \sim \frac{K(\alpha, \beta)}{2^n(n+1)!} |f^{(n+1)}(\eta)|. \tag{3.16}$$

Proof. The proof, using Theorems 2.1 and 3.3, follows essentially the same line as that of Corollary 3.2, but with the maximum of $|P_{n+1}^{(\alpha,\beta)}(x)|$ attained at x'_{n+1}. Also we have used Stirling's asymptotic formula for the gamma function [1; 6.1.39]:

$$\Gamma(an+b) \sim \sqrt{2\pi}\, e^{-an} (an)^{an+b-1/2}. \quad ▮$$

We note that when $-1<\alpha=\beta<-\frac{1}{2}$ (3.16) becomes

$$\|r_n\|_\infty \sim \frac{1}{2^{n+\alpha+1/2}(n+1)!} |f^{(n+1)}(\eta)|, \tag{3.17}$$

which agrees asymptotically with the exact form (3.11), valid for odd values of n only.

Finally, it may be shown that, for $-1<\alpha,\ \beta<-\frac{1}{2}$,

$$1<K(\alpha,\beta)<\sqrt{2}, \tag{3.18}$$

so that for large n

$$\|r_n\|_\infty<\frac{\sqrt{2}}{2^n(n+1)!}M.$$

4. Particular Cases

The theorems of the previous section are now used to derive expressions for $\|r_n\|_\infty$ in some particular cases. The results are contained in Table I. Recall that $s_n(x)$ is a truncated series of Jacobi polynomials. The first column records the name of the particular polynomial with the appropriate values of α and β. The second column gives the quantity d_n of the expression

$$\|r_n\|_\infty=d_n|f^{(n+1)}(\eta)|,$$

TABLE I

Polynomial	d_n	d_n^*
Minimax	$\frac{1}{2^n(n+1)!}$	1
Chebyshev T_n ($\alpha=\beta=-\frac{1}{2}$)	$\frac{1}{2^n(n+1)!}$	1
Legendre P_n ($\alpha=\beta=0$)	$\frac{\sqrt{\pi}}{2^{n+1}\Gamma(n+3/2)}$	$\frac{\sqrt{\pi}\,n^{1/2}}{2}$
Chebyshev U_n ($\alpha=\beta=\frac{1}{2}$)	$\frac{n+2}{2^{n+1}(n+1)!}$	$\frac{n}{2}$
Ultraspherical $C_n^{\alpha+1/2}$ ($\alpha=\beta$)		
$\alpha\geqslant-\frac{1}{2}$	$\frac{\Gamma(n/2+\alpha+1)\,\Gamma(n/2+\alpha+3/2)}{\Gamma(\alpha+1)\,\Gamma(n+\alpha+3/2)(n+1)!}$	$\frac{\sqrt{\pi}\,n^{\alpha+1/2}}{2^{2\alpha+1}\Gamma(\alpha+1)}$
$-1<\alpha<-\frac{1}{2}$, n odd	$\frac{\Gamma(n/2+\alpha+1)}{2^{n+1}\Gamma(n+\alpha+3/2)((n+1)/2)!}$	$2^{-\alpha-1/2}$
$-1<\alpha<-\frac{1}{2}$, n even		$2^{-\alpha-1/2}$
Jacobi $P_n^{(\alpha,\beta)}$ ($\alpha,\beta>-1$)		
$q=\max(\alpha,\beta)\geqslant-\frac{1}{2}$	$\frac{2^{n+1}\Gamma(n+\alpha+\beta+2)\,\Gamma(n+q+2)}{\Gamma(q+1)\,\Gamma(2n+\alpha+\beta+3)(n+1)!}$	$\frac{\sqrt{\pi}\,n^{q+1/2}}{2^{\alpha+\beta+1}\Gamma(q+1)}$
$-1<q<-\frac{1}{2}$		$K(\alpha,\beta)$

where such an expression is valid. The third column gives the quantity d_n^* of the asymptotic expression, for large n,

$$\|r_n\|_\infty \sim \frac{d_n^*}{2^n(n+1)!} |f^{(n+1)}(\eta)|.$$

The first entry in the table is for the minimax polynomial so that comparisons may be made.

It should be noted that the last entry in the d_n^* column is bounded by (3.18).

From Table I we observe that when $s_n(x)$ is a truncated series of Chebyshev polynomials of the first kind, the expression for $\|r_n\|_\infty$ is precisely the same form as that for the minimax polynomial. Also when $s_n(x)$ is a truncated series of Jacobi polynomials with $-1<\alpha, \beta<-\frac{1}{2}$, the asymptotic expression for $\|r_n\|_\infty$ is the same form as that for the minimax polynomial apart from a constant multiple which lies in the interval $(1, \sqrt{2})$.

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